A slotting technique for cross-correlation and cross-spectral density estimation for two-channel laser Doppler anemometry

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Abstract

The slotting technique for calculating the cross-correlation function and the cross-spectra from two-channel data is revisited and extended by recently developed processing steps.

1 Introduction

For calculating the correlation function or the power spectral density from randomly sampled data from laser Doppler velocity measurements, estimation procedures are required, which consider the specific characteristics of LDV data, namely the sampling of the flow velocity at random arrival times, the data noise and the correlation of the sampling rate and the instantaneous velocity. Much effort has been put onto autocorrelation and auto-spectral estimators following thee different estimator classes, slot correlation, estimating a correlation function (correlogram) from the data [4, 7, 28, 29, 11, 12, 15, 16, 20, 22, 23, 24, 25, 27], direct spectral estimators, estimating a spectrum (periodogram) directly from the randomly sampled data [4, 5, 8, 9, 10, 17, 18, 30] and interpolation methods of the randomly sampled LDV data set yielding a continuous velocity over time, which then is re-sampled equidistantly with a given sampling frequency and processed by usual signal processing tools for equidistantly sampled data, including corrections of systematic errors [2, 13, 21, 26] and noise removal [19, 21].

Much less details are known about respective estimation procedures for two-channel data yielding the cross-correlation function and the cross-spectral density. So far detailed investigations exist about the following algorithms and applications.

- In [14] the possibility to use the slotting technique for the estimation of the cross-correlation function and the cross-spectrum is mentioned. There, no weighting mechanism has been realized, no local normalization, no fuzzy slotting, and no investigation has been made about independent and dependent measurements between the channels. For autocorrelation, weighting schemes have been implemented [4], including the forward-backward inter-arrival-time weighting if transit times for individual weighting are not available [15, 16], local normalization and fuzzy slotting [28, 29, 20, 27] as well as Bessel's correction, if the data sets or data blocks are short and systematic errors due to the under-estimation of the block variance occur if the empirical block mean value is removed from the data blocks [17, 18].

- The application of the interpolation method to LDV cross-correlation and cross-spectra estimation has been investigated in [14] and [6]. Unfortunately, in both publications only special cases of two-channel measurements have been studied, namely either strictly independent or strictly coincident measurements in [6] or a mixture of only these two cases of measurements in [14]. The possibility of having a certain time delay of dependent measurements between the measurement channels has been mentioned in [14]. However, the there given procedures are valid only for a mixture of independent measurements and coincident dependent measurements between the channels, which is the case only if the respective measurement volumes of the two channels overlap. The method inherent weighting by holding the values longer if the data rate is lower can reduce the statistical bias due to the correlation between the instantaneous data rate and the velocity. At least at high data rates the suppression works efficient. Other, individual weighting schemes have not been realized for the interpolation method yet. Neither local normalization nor fuzzy slotting, originally developed for the slot correlation, have been adapted to the interpolation method so far. Bessel's correction, to suppress systematic errors due to the under-estimation of the velocity variances and velocity co-variance for short data sets or data blocks was not available at that time.

- The direct estimation has been used for the estimation of autocorrelation and auto-spectra only, including individual weighting [5, 9, 10, 30] or forward-backward inter-arrival-time weighting and Bessel's correction [17], local normalization and fuzzy time quantization [18]. Cross-correlation or cross-spectra have not been calculated with the direct estimation procedure so far.

- The direct estimation has also been used with quantized arrival times [5]. Quantized arrival times yield a quasi-equidistant data set with gaps with no data between the original samples. Filling these gaps with zeros yields an equidistant data set, which can be processed with common signal processing tools. This way either the correlation function can be calculated directly or the spectrum utilizing the fast Fourier transform. Both, the correlation function and the spectrum then are related through the Wiener-Khinchin theorem. This way the calculations for the direct estimation can be accelerated significantly. Since the time quantization changes the results obtained, this method is counted as a fourth estimation type. It has not been used previously for the calculation of the cross-correlation or the cross-spectrum.

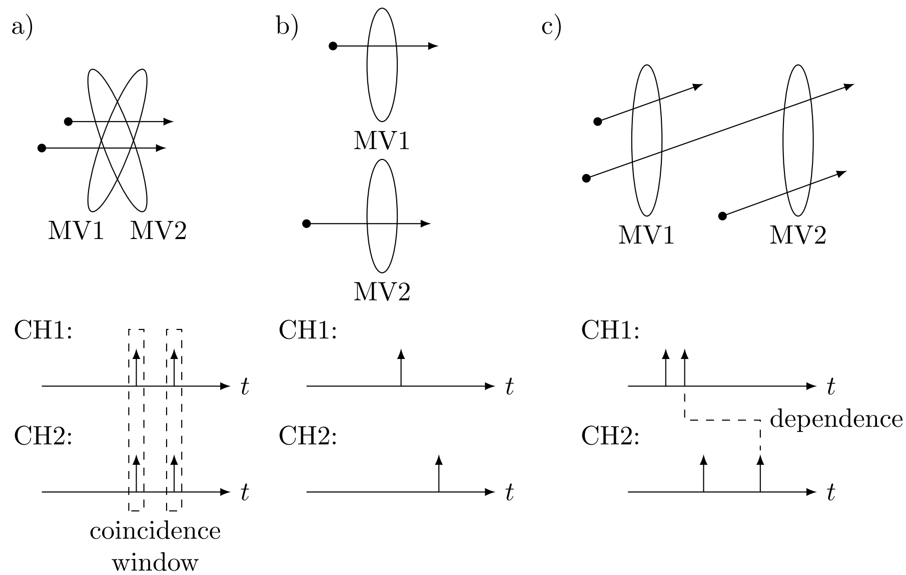

Figure 1: Fundamental sampling cases of two-channel laser Doppler data (CH1 and CH2) depending on the arrangement of the two measurement volumes MV1 and MV2: a) coincident measurements; b) independent measurements; c) mixed independent and dependent measurements with variation of the delay time between the two channels

Unfortunately, the adaption of autocorrelation and auto-spectrum estimators to the two-channel case for randomly sampled LDV data is not as straight forward as for equidistantly sampled data. While for equidistant sampling, only one of two identical data sets in the autocorrelation/auto-spectrum calculation is replaced by a second data set, besides the always present irregular sampling and the correlation between the velocity and the data rate, additionally dependent and independent samples in the two channels must be considered [14] in the LDV case. Therefore, a detailed view into adequate estimation procedures is necessary for the three classes of estimation procedures given above.

2 Dependent and independent measurements

Two-channel data from multi-component or multi-point LDV systems can produce different sampling cases depending on the configuration of the system. These sampling characteristics may lead to different systematic errors. Therefore, the following fundamental cases must be considered (Fig. 1).

- Coincident measurements: e.g. from two-component arrangements. The two channels are sampled together. This scheme yields a data set with identical sampling times $t_{1,i} = t_{2,i} = t_i$ and with identical number of samples $N_1 = N_2 = N$. This is most similar to the autocorrelation case. Estimation routines, errors and corrections are similar.

- Independent measurements: e.g. from transversal two-point measurements. The sampling of the two channels (number of samples and sampling times) is completely independent. This scheme yields a data set with no dependence between the two channels. Both, the number of samples and the sampling times of the two channels are independent of each other. The errors and corrections are different from the coincidence case.

- Mixed measurements: e.g. from two-component arrangements in free-running mode or from longitudinal two-point arrangements. There are both, independent measurements, from particles that are measured by only one of the two laser Doppler systems, and N_dep dependent measurements, from particles that cross both measurement volumes. In the latter case, the respective dependent samples $u_{\mathrm{d},1,i} = u(t_{\mathrm{d},1,i})$ and $u_{\mathrm{d},2,i} = u(t_{\mathrm{d},2,i})$ at arrival times $t_{\mathrm{d},1,i}$ and $t_{\mathrm{d},2,i}$ respectively with $i = 0 \dots N_\mathrm{dep} - 1$ are subsets of the measured data $u_{1,i} = u(t_{1,i}), i = 0 \dots N_1 - 1$ and $u_{2,j} = u(t_{2,j}), j = 0 \dots N_2 - 1$. The samples of the two channels are time delayed ($t_{\mathrm{d},1,i} \neq t_{\mathrm{d},2,i}$), where the delay t_d may vary with the instantaneous velocity.

3 The data sets

In the most general case, the data sets are assumed to be a mixture of both, independent measurements on the two channels as well as dependent measurements, which yield samples in both channels with a prefered time delay of t_d. In this, general case, two sets of irregularly sampled velocity data $u_{1,i} = u_1(t_{1,i})$ and $u_{2,j} = u_2(t_{2,j})$ at sampling times $t_{1,i}, i = 0 \dots N_1 - 1$ and $t_{2,j}, j = 0 \dots N_2 - 1$ are assumed together with individual weights $w_{1,i}$ and $w_{2,j}$ according to the velocity samples $u_{1,i}$ and $u_{2,j}$, e.g. the particle's transit times. If individual weights are not available, the inter-arrival times can be used for weighting, where both, the forward and the backward inter-arrival times may be necessary for the correlation and spectral estimations.

$$
\begin{aligned}
w_{\mathrm{bw},1,i} &= t_{1,i} - t_{1,i-1} \\
w_{\mathrm{fw},1,i} &= t_{1,i+1} - t_{1,i} \\
w_{\mathrm{bw},2,j} &= t_{2,j} - t_{2,j-1} \\
w_{\mathrm{fw},2,j} &= t_{2,j+1} - t_{2,j}
\end{aligned}
$$

To avoid that gaps in the data stream of experimental data lead to improperly large weights, as has been observed in experiments, all inter-arrival time weights derived from inter-arrival times larger than five times the mean inter-arrival time are set to zero. Due to this one looses only about 0.7 % of useful data, while the outliers of large inter-arrival times are suppressed effectively.

Two other, special cases, are considered, namely the coincident measurements, where the two channels share a common sampling, and the independent measurements, where the sampling of the two channels is completely independent, and no dependent measurements occur. In the case of only coincident measurements, the two data sets have identical sampling, leading to the velocity data $u_{1,i} = u_1(t_i)$ and $u_{2,i} = u_2(t_i)$ at sampling times $t_i, i = 0 \ldots N-1$ together with individual weights $w_{1,i}$ and $w_{2,i}$, which may be different or identical depending on the weighting scheme applied. In this case, also the forward- and backward inter-arrival times are identical for the two channels.

For completely independent sampling and inter-arrival time weighting, a differentiation of forward and backward inter-arrival times is not necessary. In this particular case, the weights can be chosen as

$$\begin{aligned} w_{\mathrm{bw},1,i} = w_{\mathrm{fw},1,i} &= t_{1,i} - t_{1,i-1} \\ w_{\mathrm{bw},2,j} = w_{\mathrm{fw},2,j} &= t_{2,j} - t_{2,j-1} \end{aligned}$$

4 Determination of the rates of dependent and independent measurements and the preffered time delay of dependent measurements

If the following procedures of correlation and spectrum estimation are used with the forward-backwards inter-arrival time weighting, then the preffered time delay t_d between the two channels is required to be determined. Therefore first, the rates of dependent and independent measurements are estimated. A similar prerequirement has been investigated in [14]. The procedure given here follows the same principals, however, the parameter identification has been modified, allowing also dependent measurements with a preferred time delay $t_\mathrm{d} \neq 0$.

Assuming a number N_1 of measurements in the first channel and N_2 in the second channel, and a number of N_dep dependent measurements, which occur in both channels, then the number of independent measurements in channel 1 is $N_1 - N_\mathrm{dep}$ and in channel 2 it is $N_2 - N_\mathrm{dep}$. Counting the number of cross-products one obtaines a total number of $N_1 N_2$ cross-products, where the N_dep dependent measurements concentrate around t_d (Fig. 2) leaving $N_1 N_2 - N_\mathrm{dep}$ cross-products with a random triangular distribution $p(\tau)$ between $-T_\mathrm{B}$ and $+T_\mathrm{B}$ following the model distribution

$$p(\tau) = A\left(1 - \frac{|\tau|}{T_\mathrm{B}}\right).$$

Defining a total length of the final correlation function $T_\mathrm{C} < 2T_\mathrm{B}$ (between $-T_\mathrm{C}/2$ and $+T_\mathrm{C}/2$), which includes all dependent measurements, one can expect to have only independent cross-products outside this interval. Note that T_C is usually chosen much smaller than the total length of the data set, if a subdivision of the data set into blocks is made, the block length should still be significantly

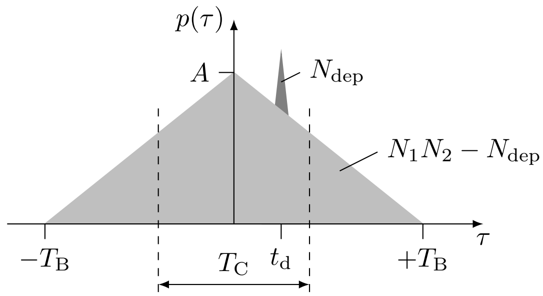

Figure 2: Dependent and independent measurements

larger then the chosen length of the correlation function. By counting the number N_{C} of cross-products falling into the interval T_{C}, assuming that the data sets of the two channels have the same duration T, the remaining $N_1 N_2 - N_{\mathrm{C}}$ cross-products distribute following the triangular shape between $-T_{\mathrm{B}}$ and $-T_{\mathrm{C}}/2$ and between $+T_{\mathrm{C}}/2$ and $+T_{\mathrm{B}}$. This yields an expected height of the triangular shape of independent measurements of

$$p\left(\pm\frac{T_{\mathrm{C}}}{2}\right) = \frac{N_1 N_2 - N_{\mathrm{C}}}{T_{\mathrm{B}} - \frac{T_{\mathrm{C}}}{2}}$$

at the time delay of $\pm T_{\mathrm{C}}/2$, which then leads to the amplitude A of the triangular shape of independent measurements

$$A = \frac{T_{\mathrm{B}}\left(N_1 N_2 - N_{\mathrm{C}}\right)}{\left(T_{\mathrm{B}} - \frac{T_{\mathrm{C}}}{2}\right)^2}.$$

All deviations from the triangular model distribution of independent measurements are interpreted as dependent measurements by the following processing methods.

To determine also the preferred time delay t_{d}, the asymmetry of the distribution of times between measurements within the interval T_{C} is used and the deviation of the number N_{C} of measured cross-products within T_{C} from the expected number of independent cross-products, yielding

$$t_{\mathrm{d}} = \frac{\sum\limits_{i=0}^{N_1-1} \sum\limits_{j=0}^{N_2-1} \left(t_{2,j} - t_{1,i}\right)_{|t_{2,j}-t_{1,i}|<T_{\mathrm{C}}/2}}{N_{\mathrm{C}} - A T_{\mathrm{C}}\left(1 - \frac{T_{\mathrm{C}}}{4T_{\mathrm{B}}}\right)}$$

The advantage of the method is that it is independent of the parameters of the following estimation of the correlation function and the spectrum, except for the time T_{C} which is re-used by all following methods as the total length of the estimated correlation function.

To determine the fraction of dependent measurements within the slots of different delay times, first the number of self- and cross-products in each slot is counted through

$$R'_{n,12}(\tau_k) = \sum_{i=0}^{N_1-1} \sum_{j=0}^{N_2-1} b_k(t_{2,j} - t_{1,i})$$

with

$$b_k(\Delta t) = \begin{cases} 1 & \text{for } |\Delta t - k\Delta\tau| < {}^{\Delta\tau}/_{2} \\ 0 & \text{otherwise} \end{cases} .$$

This number is compared with the expected number of products if only independent measurements are assumed, given through

$$\frac{A}{F}\left(1 - \left|\frac{\tau_k}{T_\mathrm{B}}\right|\right)$$

and finally, the fraction of dependent measurements in each slot is derived as

$$d(\tau_k) = 1 - \frac{\frac{A}{F}\left(1 - \left|\frac{\tau_k}{T_\mathrm{B}}\right|\right)}{R'_{n,12}(\tau_k)}.$$

This number is (around) 0, if only independent measurements are taken and, it is (around) 1, if only dependent measurements are counted in the respective slot. The latter case cannot be achieved due to the fact, that every two different measurements are independent, regardless of the kind of each of the two measurements.

5 Slotting Technique

The slotting technique for the estimation of the cross-correlation function has been given as a reference algorithm in [14]. The algorithm presented here, principally follows the procedure, except for a few extensions/modifications, which are

1. Individual weighting (e.g. transit-time weighting) has been implemented as well as forward-backward inter-arrival time weighting [15, 16, 17] for the case that reliable values of transit times are not available. The latter weighting scheme additionally requires the estimation of the prefered time delay t_d between the measurement channels as given above.

2. If a large gap between measurements occurs, as has been seen in experimental data, the forward-backward inter-arrival time weighting factor is set to zero to suppress the affecting influence to the derived statistical functions.

3. In the case of mixed dependent and independent measurements, especially if there is a certain delay of dependent measurements between the

channels, then the unique identification of the dependent measurements among the samples in the two channels has not been realized so far. If both, the dependent and the independent measurements, are mixed in the summations, then the weighting of the two kinds of measurements get incorrect. While independent measurements are correctly weighted by the product of the indovidual weights, the dependent measurements get overweighted in this case. Dependent measurements should be better weighted by the square root of the product of the two individual weighting factors. However, within a summation, variations of the weighting factors are not possible. Therefore, for each slot, the fraction of dependent measurements $d(\tau_k)$ is used to obtain an exponent $\gamma(\tau_k)$ for the product of weighting factors, which decreases with increasing number of dependent measurements.

$$\gamma(\tau_k) = 1 - \frac{d(\tau_k)}{2}$$

4. Bessel's correction of the correlation estimate is added, which suppresses systematic deviations due the under-estimation of the velocity variance for short data sets, if the mean is estimated and removed from the data sets following [17].

Since these little modifications influence the entire estimation procedure, it is summarized here including the appropriate changes.

5.1 Data pre-processing

The available data may be subdivided into blocks of a certain time duration T_B or the data may be obtained in blocks of a given record length. Due to the combination of Bessel's correction and the temporal limitation of the correlation function, both given below, the block duration can be chosen very flexible (compare [17]). It should be larger than the expected correlation interval of the flow and can be as large as the full data set.

From the data series $u_{1,i} = u_1(t_{1,i}), i = 0 \dots N_1 - 1$ and $u_{2,i} = u_2(t_{2,i}), i = 0 \dots N_2 - 1$ and the appropriate weigths $w_{1,i}$ and $w_{2,i}$, e.g. the transit times, one can calculate the block mean values as

$$\begin{aligned}
\bar{u}_1 &= \frac{\sum\limits_{i=0}^{N_1-1} w_{1,i} u_{1,i}}{\sum\limits_{i=0}^{N_1-1} w_{1,i}} \\
\bar{u}_2 &= \frac{\sum\limits_{i=0}^{N_2-1} w_{2,i} u_{2,i}}{\sum\limits_{i=0}^{N_2-1} w_{2,i}}
\end{aligned}$$

or, using the backward inter-arrival times as given above

$$\bar{u}_1 = \frac{\sum\limits_{i=0}^{N_1-1} w_{\mathrm{bw},1,i} u_{1,i}}{\sum\limits_{i=0}^{N_1-1} w_{\mathrm{bw},1,i}}$$

$$\bar{u}_2 = \frac{\sum\limits_{i=0}^{N_2-1} w_{\mathrm{bw},2,i} u_{2,i}}{\sum\limits_{i=0}^{N_2-1} w_{\mathrm{bw},2,i}}$$

which is identical to the classical arrival-time weighting [3]. If the inter-arrival time between two samples exceeds a certain limit, the inter-arrival time factor can be set zero. Good experience has been obtained with a maximum value of five times the mean inter-arrival time. Due to this one looses about 0.7 % of useful data, while the outliers of large inter-arrival times are suppressed effectively.

The block mean values then can be removed from the data to generate mean free data blocks for the following calculations of the cross-correlation function and the appropriate power spectral density.

5.2 Estimation of the initial correlation functions

The two correlation functions of the weighted velocities and that of the weights are derived for individual weighting (e.g. transit-time weighting) as

$$R'_{u,12}(\tau_k) = \sum_{i=0}^{N_1-1} \sum_{j=0}^{N_2-1} (w_{1,i} w_{2,j})^{\gamma(\tau_k)} u_{1,i} u_{2,j} b_k(t_{2,j} - t_{1,i})$$

$$R'_{w,12}(\tau_k) = \sum_{i=0}^{N_1-1} \sum_{j=0}^{N_2-1} (w_{1,i} w_{2,j})^{\gamma(\tau_k)} b_k(t_{2,j} - t_{1,i})$$

$$R''_{w,12}(\tau_k) = \sum_{i=0}^{N_1-1} \sum_{j=0}^{N_2-1} w_{1,i} w_{2,j} b_k(t_{2,j} - t_{1,i})$$

with $b_k(\Delta t)$ and $\gamma(\tau_k)$ as introduced above, or, for forward-backward inter-arrival weighting with only independent measurements ($d(\tau_k) \equiv 0$ leading to $\gamma(\tau_k) \equiv 1$)

$$R'_{u,12}(\tau_k) = \sum_{i=0}^{N_1-1} \sum_{j=0}^{N_2-1} w_{\mathrm{bw},1,i} w_{\mathrm{bw},2,j} u_{1,i} u_{2,j} b_k(t_{2,j} - t_{1,i})$$

$$R'_{w,12}(\tau_k) = \sum_{i=0}^{N_1-1} \sum_{j=0}^{N_2-1} w_{\mathrm{bw},1,i} w_{\mathrm{bw},2,j} b_k(t_{2,j} - t_{1,i})$$

$$R''_{w,12}(\tau_k) = R'_{w,12}(\tau_k)$$

or, for forward-backward inter-arrival weighting with a mixture of independent and dependent measurements

$$R'_{u,12}(\tau_k) = \sum_{i=0}^{N_1-1}\sum_{j=0}^{N_2-1}\left\{\begin{array}{ll}(w_{\mathrm{bw},1,i}w_{\mathrm{fw},2,j})^{\gamma(\tau_k)} & \text{if } t_{1,i}<t_{2,j}-t_\mathrm{d}\\ (w_{\mathrm{fw},1,i}w_{\mathrm{bw},2,j})^{\gamma(\tau_k)} & \text{if } t_{1,i}>t_{2,j}-t_\mathrm{d}\end{array}\right\} u_{1,i}u_{2,j}b_k(t_{2,j}-t_{1,i})$$

$$R'_{w,12}(\tau_k) = \sum_{i=0}^{N_1-1}\sum_{j=0}^{N_2-1}\left\{\begin{array}{ll}(w_{\mathrm{bw},1,i}w_{\mathrm{fw},2,j})^{\gamma(\tau_k)} & \text{if } t_{1,i}<t_{2,j}-t_\mathrm{d}\\ (w_{\mathrm{fw},1,i}w_{\mathrm{bw},2,j})^{\gamma(\tau_k)} & \text{if } t_{1,i}>t_{2,j}-t_\mathrm{d}\end{array}\right\} b_k(t_{2,j}-t_{1,i})$$

$$R''_{w,12}(\tau_k) = \sum_{i=0}^{N_1-1}\sum_{j=0}^{N_2-1}\left\{\begin{array}{ll}w_{\mathrm{bw},1,i}w_{\mathrm{fw},2,j} & \text{if } t_{1,i}<t_{2,j}-t_\mathrm{d}\\ w_{\mathrm{fw},1,i}w_{\mathrm{bw},2,j} & \text{if } t_{1,i}>t_{2,j}-t_\mathrm{d}\end{array}\right\} b_k(t_{2,j}-t_{1,i})$$

If the forward or backward inter-arrival time between two samples exceeds a certain limit, the weigthing factor should be set zero. Good experience has been obtained with a maximum value of five times the mean inter-arrival time. Due to this one looses about 0.7 % of useful data, while the outliers of large inter-arrival times are suppressed effectively. Note, that appropriate pre-factors for the correlation functions have not been given, since the following normalization doesn't require an appropriate normalization here. Note further, that the maximum time lag of the slot-correlation estimator is typically chosen smaller than the duration of the measurement. With a chosen temporal resolution of the correlation function $\Delta\tau$ and a number of samples K, the correlation function will be estimated for $k = -\lfloor K/2\rfloor \ldots \lfloor (K-1)/2\rfloor$ spanning a total length of the correlation function of $T_\mathrm{C} = K\Delta\tau$. The same T_C is used to estimate the rates of independent and dependent measurements and the preffered time delay between the channels t_d, which is used within the correlation estimation for forward-backward interarrival-time weighting to obtain independent weighting factors for all measurements, dependent as well as independent.

Also note, that the above sums include the dependent pairs of measurements, which have different sampling probabilities than the pairs of independent measurements, like self- and cross-products in the autocorrelation case. Ideally, all cross-procucts of dependent measurements should be removed from the double sums ($R'_{u,12}$ and $R'_{w,12}$). For coincident measurements, the identification of dependent measurements can be obtained through the common sampling of the two data channels. In this particular case, the removal of the dependent measurements can be done equivalently to the autocorrelation case. Unfortunately, no way of identifying dependent measurements uniquely from mixed independent and dependent measurements has been found so far, where the dependent measurements may have a temporal delay between the two channels, which may even vary depending on the instantaneous velocity. Therefore, this influence is present in the general estimation procedure. However, since any noise in the two channels is independent, and varying sampling probabilities are corrected by the following normalization, no systematic error is expected due to this influence. Furthermore, this influence decreases with an increasing length of the data set or data block. Since the effort of calculations increases linearly with the length of the data set, the slotting technique can always be used with the maximum length of the data sets without additional block subdevision.

5.3 Normalization, Bessel's correction and final transformation

The final estimate of the correlation function is obtained by normalization as

$$R_{12}(\tau_k) = \frac{R'_{u,12}(\tau_k)}{R'_{w,12}(\tau_k)} + c_B$$

including Bessel's correction, where the correction c_B is related to the estimated variances of the mean estimators above. In the case of the slotting technique c_B is obtained similar to the procedure given in [17] adpated to the cross-correlation case as

$$c_B = \frac{\sum\limits_{k=-\lfloor K/2 \rfloor}^{\lfloor (K-1)/2 \rfloor} R'_{u,12}(\tau_k)}{W - \sum\limits_{k=-\lfloor K/2 \rfloor}^{\lfloor (K-1)/2 \rfloor} R''_{w,12}(\tau_k)}.$$

with

$$W = \left(\sum_{i=0}^{N_1-1} w_{1,i}\right)\left(\sum_{j=0}^{N_2-1} w_{2,j}\right)$$

for the case of individual weights (e.g. transit-time weighting) or

$$\begin{aligned}
W &= \sum_{i=0}^{N_1-1}\sum_{j=0}^{N_2-1} \left\{ \begin{array}{ll} w_{\mathrm{bw},1,i}w_{\mathrm{fw},2,j} & \text{if } t_{1,i} < t_{2,j} - t_\mathrm{d} \\ w_{\mathrm{fw},1,i}w_{\mathrm{bw},2,j} & \text{if } t_{1,i} > t_{2,j} - t_\mathrm{d} \end{array} \right\} \\
&= \sum_{i=0}^{N_1-1} \sum_{\substack{j=0 \\ t_{2,j}<t_{1,i}+t_\mathrm{d}}}^{N_2-1} w_{\mathrm{fw},1,i}w_{\mathrm{bw},2,j}u_{1,i}u_{2,j} + \sum_{i=0}^{N_1-1} \sum_{\substack{j=0 \\ t_{2,j}>t_{1,i}+t_\mathrm{d}}}^{N_2-1} w_{\mathrm{bw},1,i}w_{\mathrm{fw},2,j}u_{1,i}u_{2,j} \\
&= \sum_{i=0}^{N_1-1} w_{\mathrm{fw},1,i}u_{1,i}\left(\sum_{\substack{j=0 \\ t_{2,j}<t_{1,i}+t_\mathrm{d}}}^{N_2-1} w_{\mathrm{bw},2,j}u_{2,j}\right) + \sum_{j=0}^{N_2-1} w_{\mathrm{fw},2,j}u_{2,j}\left(\sum_{\substack{i=0 \\ t_{1,i}<t_{2,j}-t_\mathrm{d}}}^{N_1-1} w_{\mathrm{bw},1,i}u_{1,i}\right)
\end{aligned}$$

in the case of forward-backward inter-arrival weighting.

The final cross-correlation estimate can then be transformed by means of the discrete Fourier transform (DFT) to a power spectral density

$$S_{12}(f_j) = \Delta\tau \mathrm{DFT}\left\{R_{12}(\tau_k)\right\} = \Delta\tau \sum_{k=-\lfloor K/2 \rfloor}^{\lfloor (K-1)/2 \rfloor} R_{12}(\tau_k)\mathrm{e}^{-2\pi \mathrm{i} f_j \tau_k}$$

with $f_j = j\Delta f, j = -\lfloor K/2 \rfloor \dots \lfloor (K-1)/2 \rfloor$ giving a frequency resolution of $\Delta f = 1/K\Delta\tau$.

5.4 Remarks

In the autocorrelation case, different probability densities for self and cross products and the fact of systematic errors of self-products due to noise in the data make it necessary to exclude self-products from the sums of the slot-correlation estimation [22]. In the cross-correlation case the noise in the two channels is independent and hence, no systematic errors occur in the correlation due to this noise. A statistical bias due to the correlation of the velocity and the instantaneous data rate are suppressed due to the implementation of the weighting schemes [4, 15, 16, 17]. An example program can be found at [1] including local normalization and fuzzy slotting [28, 29, 20, 27, 18] adapted to the cross-correlation case. Since dependent measurements are counted, a modification of the weighting exponents depending on the fraction of dependent measurements in a given slot is neccessary to avoid systematic errors due to different probabilities of dependent and independent measurements.

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